A High-Level Overview of Cryptography

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- The word has its origin in greek[1]:
  - κρυπτός (*kryptos*) meaning hidden[2].
  - γράφος (*graphos*) meaning writing[3].
- The area has been around for ages.
- We should not confuse it with *steganography*.
- Steganography concerns hiding a message's *existence*.
- Cryptography concerns hiding a message's *contents*.

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- Then it was an art, now it's a science.
- People used 'clever' constructions.
- These were thought to be secure: 'How can anyone figure this out?'
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A quote[4]

[A cryptosystem] should not require secrecy, and it should not be a problem if it falls into the enemy hands;

Kerckhoff's Principle

- No security-by-obscurity
- The key should be the only secret

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Remark

- This doesn't mean we must tell the adversary what we're using.
- But we shouldn't loose any security if we do.

Idea

- Alice and Bob share a (small) common secret.
- Alice takes a message, combines it with the secret, sends it to Bob.
- If Eve captures the whatever Alice sent, she shouldn't learn anything about the message.
- Bob combines what he received with the secret and gets the message.

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Block-cipher encryption

Input A fixed-sized *key* k, a fixed-sized block of *plaintext* p.

Output A fixed-sized block of *ciphertext* c.

Notation $\mathrm{Enc}_k(p) = c$

Block-cipher decryption

Input A fixed-sized *key* k, a fixed-sized block of *ciphertext* c.

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Definition (Crypto system[5])

- A *crypto system* is a tuple $(\mathcal{M}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where:
  - $\mathcal{M}$ is a finite set of *plaintexts* or messages,
  - $\mathcal{C}$ is a finite set of *ciphertexts*,
  - $\mathcal{K}$ is the *keyspace*, a finite set of keys.
  - $\mathcal{E}$ and $\mathcal{D}$ are the sets of encryption and decryption rules, respectively.
- For every $k \in \mathcal{K}$ there is a $\mathrm{Enc}_k \in \mathcal{E}$ and $\mathrm{Dec}_k \in \mathcal{D}$ such that
  - $\mathrm{Enc}_k : \mathcal{M} \to \mathcal{C}$ and $\mathrm{Dec}_k : \mathcal{C} \to \mathcal{M}$ are functions and
  - $\mathrm{Dec}_k(\mathrm{Enc}_k(m)) = m$ for all plaintexts $m \in \mathcal{M}$.

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Definition (Shift Cipher)

- Let $\mathcal{M} = \mathcal{C} = \mathcal{K} = \mathbb{Z}_{29}$
- For each $k \in \mathcal{K}$ we define

$$\begin{aligned} \text{Enc}_k(m) &= (m + k) \bmod 29, m \in \mathcal{M}, \text{ och} \\ \text{Dec}_k(c) &= (c - k) \bmod 29, c \in \mathcal{C}\,. \end{aligned}$$

Example

- $\text{Enc}_3(7) = 7 + 3 \bmod 29 = 10$ h→J
- $\text{Enc}_3(4) = 4 + 3 \bmod 29 = 7$ e→G
- $\text{Enc}_3(9) = 9 + 3 \bmod 29 = 12$ j→L

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Exercise

- What do we have to do to set this up between two parties, say Alice and Bob?
- What problems do we have to solve?

Definition (Perfect secrecy[6])

- Cryptosystem $(\mathcal{M}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$.
- Stochastic variables M, C.
- *Perfect secrecy* if and only if

$$\Pr(M = m \mid C = c) = \Pr(M = m)$$

for all $m \in \mathcal{M}$ and $c \in \mathcal{C}$.

Note

Equivalent to $H(M \mid C) = H(M)$, i.e. ciphertext does not reveal anything about plaintext.

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Theorem (Shannon's theorem)

- *Assume cryptosystem* $(\mathcal{M}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$ *such that* $|\mathcal{K}| = |\mathcal{C}| = |\mathcal{M}|$.
- *This provides perfect secrecy if and only if*
  1. *every key* $k \in \mathcal{K}$ *is used with equal probability* $1/|\mathcal{K}|$,
  2. *for every plaintext* $m \in \mathcal{M}$ *and* $c \in \mathcal{C}$ *there is a unique key such that* $\text{Enc}_k(m) = c$.

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Example (One-time Pad)

- Let n be a positive integer.
- Let $\mathcal{M} = \mathcal{C} = \mathcal{K} = (\mathbb{Z}_2)^n$.
- For each key $k = (k_1, \ldots, k_n) \in \mathcal{K}$, plaintexts $m = (m_1, \ldots, m_n) \in \mathcal{M}$ and ciphertexts $c = (c_1, \ldots, c_n) \in \mathcal{C}$ we define

$$\mathrm{Enc}_k(m) = (m_1 + k_1, \ldots, m_n + k_n)$$

- We also define Dec = Enc.
- $k \in \mathcal{K}$ must be chosen uniformly randomly for each encryption.

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Definition (Pseudo-random permutation, PRP[7])

- Let $F\colon \{0,1\}^s \times \{0,1\}^n \to \{0,1\}^n$.
- F is a PRP if
  1. for any $k \in \{0,1\}^s$, F is a bijection;
  2. for any $k \in \{0,1\}^s$, we can 'efficiently' evaluate $F_k(x)$;
  3. for all 'efficient' distinguishers D,

$$\left|\Pr[D^{F_k}(1^n) = 1] - \Pr[D^{f_n}(1^n) = 1]\right| < \epsilon(s)$$

  if we choose $k \in \{0,1\}^s$ and the random permutation f_n uniformly at random.

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Idea

- We want a function which we can efficiently compute.
- However, it shouldn't be possible to find its inverse.

Example

Easy $f(x) = y$

Hard $f^{-1}(y) = x$

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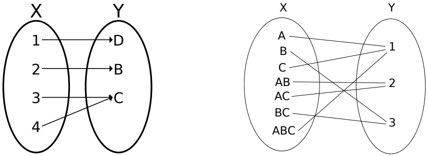

(a) $h\colon X \to Y$ (b) $h'\colon X \to Y$

Figure: Two non-injective, surjective functions h and h'.

Exercise

Could either of these two functions be one-way functions?

Definition (One-way function[8])

- Let $h\colon \{0,1\}^* \to \{0,1\}^*$.
- h is *one-way* if
  1. there exists an efficient algorithm A such that $A(x) = h(x)$;
  2. for every efficient algorithm A', every positive polynomial $p(\cdot)$ and all sufficiently large n's

$$\Pr\left[A'(h(x), 1^n) \in h^{-1}(h(x))\right] < \frac{1}{p(n)}$$

[8] Oded Goldreich. *Foundations of cryptography, Vol. 1: Basic tools.* Cambridge: Cambridge Univ. Press, 2001. ISBN: 0-521-79172-3.

Example (Implementations you might've heard of)

- MD5
- SHA1
- SHA256 (SHA-2)
- SHA-3

Example (Applications)

- Verifying file content integrity
- Digital signatures
- Protect passwords

Example

- Let $\text{Enc}_k(\cdot) = \text{Dec}_k(\cdot) = \cdot \oplus k \bmod 2$.
- Alice and Bob share k.
- Alice sends $\text{Enc}_k(m) = c$ to Bob.
- Eve intercepts c, she cannot get to m.
- Eve computes $c' = c \oplus m_E$ and passes c' to Bob.
- Bob computes $\text{Dec}_k(c') = \text{Dec}_k(c \oplus m_E) = m \oplus k \oplus m_E \oplus k = m \oplus m_E$.

Exercise

How can we solve this? Bob needs to know that Eve modified the message!

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Idea: MACs

- Alice and Bob need something that Eve doesn't know how to modify.
- If that something is tied to the message, then a modified message would be detectable.

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Any ideas on how we can construct such a thing?

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Example

- Let h be a one-way function.
- If we use $h(c) = t$, then Eve can also compute the hash function: $h(c') = t'$.
- A secret hash function would violate Kerckhoff's principle, so that's not an option.
- If we instead use the message, rather than the ciphertext.
- Then $h(m) = t$ and
  - $\text{Dec}_k(c') = m' = m \oplus m_E, h(m') \neq t$.
  - $\text{Dec}_k(c) = m, h(m) = t$.
- Eve cannot compute the hash function, she doesn't have m!
  - Bob: But neither do I!

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- If we use $h(c) = t$, then Eve can also compute the hash function: $h(c') = t'$.
- A secret hash function would violate Kerckhoff's principle, so that's not an option.
- If we instead use the message, rather than the ciphertext.
- Then $h(m) = t$ and
  - $\mathrm{Dec}_k(c') = m' = m \oplus m_E, h(m') \neq t$.
  - $\mathrm{Dec}_k(c) = m, h(m) = t$.
- Eve cannot compute the hash function, she doesn't have m!
  - Bob: But neither do I!

Solution

- *Let s be a secret shared between Alice and Bob.*
- $h(c \parallel s) = t$, *Eve doesn't know* s.
- *Bob can immediately check* $h(c' \parallel s) \neq t$.

Remark

- It requires even a bit more than this!
- But the idea is correct.

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Solution (Hash-based message-authentication code, HMAC[9])

- *Let h be a one-way function.*
- *Let c be the ciphertext, s our MA secret.*
- *Then tag $t = \mathrm{HMAC}_s(c)$, where*

$$\mathrm{HMAC}_s(c) = h\left[(s \oplus p_o) \parallel h\left[(s \oplus p_i) \parallel c\right]\right],$$

and p_i, p_o are inner and outer pads, respectively.

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This is proven secure in [9]!

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- If we could do it just before we use them?

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- What if we could securely exchange keys at a distance?
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Solution (Requirements)

- *We need a problem that is easy for Alice and Bob.*
- *It should be hard for Eve.*

Definition (Discrete Logarithm Problem, DLP)

- Let $\mathbb{Z}_p^*$ be the multiplicative group of residues modulo $p \in \mathbb{N}$, where p is a prime.

$$\begin{aligned} \text{Given} \quad & g, g^x \in \mathbb{Z}_p^* \\ \text{Find} \quad & x. \end{aligned}$$

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Given $g, g^x, g^y, g^z \in \mathbb{Z}_p^*$

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Exercise

- Diffie and Hellman[11] used DHP to create a key-exchange protocol.
- Take some time to figure out how we can use these problems to achieve what we want.

Reminder

- Alice and Bob want to exchange a secret key.
- Then they can use the key to encrypt their communications.

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Definition (Diffie-Hellman key-exchange)

- Let $g \in \mathbb{Z}_p^*$ (publicly known, e.g. RFC, standard dots).
- Alice generates random $0 < x < |\mathbb{Z}_p^*|$.
- She send g^x to Bob.
- Bob generates random $0 < y < |\mathbb{Z}_p^*|$.
- He sends g^y to Alice.
- Alice has x and g, g^y.
- Bob has g, g^x and y.
- They both compute $g^{xy} = (g^y)^x = (g^x)^y$.
- Eve has g, g^x, g^y.
- By DHP she cannot compute g^{xy}.

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Idea

- Fine, we can use g^{xy} as a key in a cipher.
  - $\mathrm{Enc}_{g^{xy}}(m)$, where Enc is a symmetric cipher.
- But shouldn't we be able to include a message directly?

Definition (ElGamal Encryption Scheme[12])

Set-up:

- Let $g \in \mathbb{Z}_p^*$, randomly choose $0 < x < |\mathbb{Z}_p^*|$.
- Alice publishes $\mathbb{Z}_p^*, g, g^x$ to everyone.

Encryption:

- Bob chooses random $0 < y < |\mathbb{Z}_p^*|$ and computes g^y.
- Bob's message $m \in \mathbb{Z}_p^*$.
- He sends $(g^y, m(g^x)^y)$ to Alice.

Decryption:

- Alice computes $(g^y)^x$ and $m(g^x)^y((g^y)^x)^{-1} = m$.

[12] Taher ElGamal. 'A Public Key Cryptosystem and a Signature Scheme Based on Discrete Logarithms'. In: *Advances in Cryptology: Proceedings of CRYPTO 84*. Ed. by George Robert Blakley and David Chaum. Berlin, Heidelberg: Springer Berlin Heidelberg, 1985, pp. 10–18. ISBN:

Idea

- Sure, if Bob sends a message to Alice, he's sure she's the only one who can decrypt it.
- Can't we turn this around?
  - Can't Alice use the same system to ensure Bob knows the message came from Alice?

Exercise

- Look at the ElGamal encryption scheme for a bit.
- Try to find a way to 'run it backwards'.

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Definition (ElGamal Signature Scheme[13])

Set-up:

- Let $g \in \mathbb{Z}_p^*$ and $\boxed{h \text{ be a one-way function}}$.
- Alice publishes $\mathbb{Z}_p^*, g, g^x$ to everyone.

Signing $m \in \mathbb{Z}_p^*$:

- $\boxed{\text{Alice}}$ chooses random $0 < y < |\mathbb{Z}_p^*|$ and computes $r = g^y \in \mathbb{Z}_p^*$.
- She computes $\boxed{s = (h(m) - xr)y^{-1} \pmod{|\mathbb{Z}_p^*|}}$.
- She sends (r, s) to Bob.

Verification:

- $\boxed{\text{Bob checks if } g^{h(m)} \stackrel{?}{=}_{\mathbb{Z}_p^*} (g^x)^r r^s} =_{\mathbb{Z}_p^*} (g^x)^{g^y} (g^y)^{(h(m)-xg^y)y^{-1}} =_{\mathbb{Z}_p^*} g^{xg^y + h(m) - xg^y}$

Remark

- It works without the hash.
- But then we can multiply two messages and still get a valid signature.

Definition (Homomorphism)

A *homomorphism* is a map (function) that preserves structure between two algebraic structures.

Example

- Let $G_1 = (\mathbb{R}, \cdot)$ and $G_2 = (\mathbb{R}, +)$ be groups.
- $g_1, g_1' \in G_1$ and $g_2, g_2' \in G_2$.
- Consider $\log : G_1 \to G_2$.
- $\log(g_1 \cdot g_1') = g_2 + g_2'$.

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Exercise

The encryption (decryption) function of the ElGamal cryptosystem is a homomorphism, what structure does it preserve?

Example (ElGamal's homomorphism)

- Messages m, m', ciphertexts $(g^y, m \cdot g^{xy}), (g^{y'}, m' \cdot g^{xy'})$.
- Remember: private key x, hence the same.
- Create ciphertext

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\begin{aligned}
(g^y g^{y'}, m \cdot g^{xy} \cdot m' \cdot g^{xy'}) &= (g^{y+y'}, m \cdot m' \cdot g^{xy+xy'}) \\
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- Decryption: take $g^{y+y'}$, compute $(g^{y+y'})^x = g^{x(y+y')}$.
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Remark

- We use a hash function in the signature scheme to counter the homomorphic property.
- $h(m) \cdot h(m') \neq h(m \cdot m')$.
- Without the hash function we could create a valid signature for a new message *without knowing the signature key!*

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- There are many schemes with different homomorphic properties.
- There is even *fully homomorphic encryption* [12].

Example (Yao's millionaires' problem)

- Two millionaires meet in the street.
- They want to find out who is the richer.
- However, they don't want to reveal how many millions they each have.

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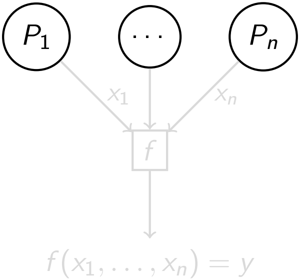
Idea

- We have n participants $P_1, \ldots, P_n$.
- Each person has a *secret* input value x_j for $1 \leq j \leq n$.
- But they desperately want to know $y = f(x_1, \ldots, x_n)$.



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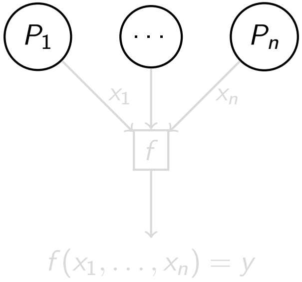

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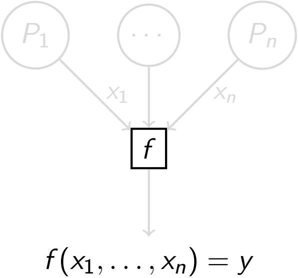

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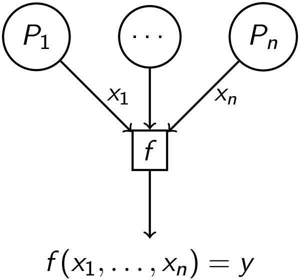

Example (Trivial solution)

- The n participants $P_1, \ldots, P_n$ agree on a trusted third-party (a TTP).
- Each participant give their secret to the TTP.
- The TTP trusted third-party performs the computation.
- Every participant receives the result from the TTP.

Definition (Secure multiparty computation, MPC)

- n participants $P_1, \ldots, P_n$.
- n secret inputs $x_1, \ldots, x_n$.
- A protocol π is executed by the participants.
- At the end of the protocol each participant learns $y = f(x_1, \ldots, x_n)$.
- The participants executing π should be *equivalent* to giving $x_1, \ldots, x_n$ to a TTP T who computes $f(x_1, \ldots, x_n) = y$ and returns y to each participant.

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- We can construct protocols for arbitrary functions f.
- Efficiency varies though.
- However, there are practically feasible protocols.
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Example (Sugar beet auctions[14])

- Several thousand farmers produce sugar beets.
- These are sold to the monopoly Danisco, the sugar producer.
- Contracts are allocated via a nation-wide exchange, a double auction.
- A double auction contains multiple sellers and multiple buyers.
- The purpose is to find the *market clearing price*.

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- Each buyer places a bid specifying how much he is willing to buy *at each potential price*.
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- Alice must prove her identity to Eve.
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- Eve asks Alice to sign the message m, if the signature verifies under the public key Eve believes Alice.

Gaaahh!

- Now Eve can show this message (chosen by Eve) with Alice's signature on it!
- What if Eve's chosen message was 'I give all my money to Eve'?

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- What if Eve's chosen message was 'I give all my money to Eve'?

Example

- Alice must prove her identity to Eve.
- Eve has Alice's public key, and knows it belongs to Alice.
- Alice wants to prove she is the owner of the private key belonging to the public key that Eve has.
- Eve asks Alice to sign the message m, if the signature verifies under the public key Eve believes Alice.

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Idea

- Alice wants to prove that she knows the discrete logarithm x of a value g^x.
- She will do this without revealing x to Eve.

Definition (Schnorr's protocol[15])

- Prover wants to prove knowledge of x for $g^x = y$.
- Prover commits to randomness r, by sending $t = g^r$.
- Verifier replies with randomly chosen challenge c.
- After receiving c, prover replies with $s = r + cx$.
- Verifier accepts if $g^s = g^{r+cx} = g^r(g^x)^c = ty^c$.

[15] C. P. Schnorr. 'Efficient signature generation by smart cards'. In: *Journal of Cryptology* 4.3 (1991), pp. 161–174. ISSN: 1432-1378. DOI: 10.1007/BF00196725. URL: http://dx.doi.org/10.1007/BF00196725.

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Proof outline.

- We need to prove *completeness*: for all (most) statements the verifier will accept.
- We need to prove *soundness*: for all (most) false statements the verifier will reject.
- We need to prove that it is zero-knowledge.

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□

Zero-knowledge

- Transcript for protocol: (t, c, s).
- Probability for transcript occurring: $\frac{1}{|R|} \cdot \frac{1}{\deg g}$.
- Simulate protocol: randomly choose c, randomly choose s, compute t by $g^s y^c$.
- We see that we get the same probability distribution.
- Thus the simulated transcripts are indistinguishable from the real ones.

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