Applied Information Theory

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1. Introduction
   - History
2. Shannon entropy
   - Definition of Shannon Entropy
   - Properties for Shannon entropy
   - Conditional entropy
   - Information density and redundancy
   - Information gain
3. Application in security
   - Passwords
   - Research about human chosen passwords
   - Identifying information

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- Created 1948 by Shannon's paper 'A Mathematical Theory of Communication' [Sha48].
- He starts using the term 'entropy' as a measure for information.
  - In physics entropy measures the disorder of molecules.
  - Shannon's entropy measures disorder of information.
- He used this theory to analyse communication.
  - What are the theoretical limits for different channels?
  - How much redundancy is needed for certain noise?

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- It's also interesting for security.
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  - 'Efficiency' of passwords
  - Measure identifiability
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Definition (Shannon entropy)

- Stochastic variable $\mathbf{X}$ assumes values from X.
- Shannon entropy $H(\mathbf{X})$ defined as

$$H(\mathbf{X}) = -K \sum_{x \in X} \Pr(\mathbf{X} = x) \log \Pr(\mathbf{X} = x),$$

- Usually $K = \frac{1}{\log 2}$ to give entropy in unit bits (bit).

Shannon entropy can be seen as . . .

- . . . how much choice in each event.
- . . . the uncertainty of each event.
- . . . how many bits to store each event.
- . . . how much information it produces.

Example (Toss a coin)

- Stochastic variable $\mathbf{S}$ takes values from $S = \{h, t\}$.
- We have $\Pr(\mathbf{S} = h) = \Pr(\mathbf{S} = t) = \frac{1}{2}$.
- This gives $H(\mathbf{S})$ as follows:

$$
\begin{aligned}
H(\mathbf{S}) &= -\left(\Pr(\mathbf{S} = h) \log \Pr(\mathbf{S} = h) + \Pr(\mathbf{S} = t) \log \Pr(\mathbf{S} = t)\right) \\
&= -2 \times \frac{1}{2} \log \frac{1}{2} = \log 2 = 1.
\end{aligned}
$$

Example (Roll a die)

- Stochastic variable $\mathbf{D}$ takes values from $D = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$.
- We have $\Pr(\mathbf{D} = d) = \frac{1}{6}$ for all $d \in D$.
- The entropy $H(\mathbf{D})$ is as follows:

$$
\begin{aligned}
H(\mathbf{D}) &= -\sum_{d \in D} \Pr(\mathbf{D} = d) \log \Pr(\mathbf{D} = d) \\
&= -6 \times \frac{1}{6} \log \frac{1}{6} = \log 6 \approx 2.585.
\end{aligned}
$$

Remark

- If we didn't know already, we now know that a roll of a die . . .
  - contains more 'choice' than a coin toss.
  - is more uncertain to predict than a coin toss.
  - requires more bits to store than a coin toss.
  - produces more information than a coin toss.
- What if we modify the die a bit?

Example (Roll of a modified die)

- Stochastic variable D' taking values from D.
- We now have $\Pr(\mathbf{D}' = ⚅) = \frac{9}{10}$ and $\Pr(\mathbf{D}' = d) = \frac{1}{10} \times \frac{1}{5}$ for $d \neq$ ⚅.
- This yields

$$
\begin{aligned}
H(\mathbf{D}') &= -\left(\frac{9}{10} \log \frac{9}{10} + \sum_{d \neq 6} \frac{1}{50} \log \frac{1}{50} \right) \\
&= -\frac{9}{10} \log \frac{9}{10} - 5 \times \frac{1}{50} \log \frac{1}{50} \\
&= -\frac{9}{10} \log \frac{9}{10} - \frac{1}{10} \log \frac{1}{50} \approx 0.486.
\end{aligned}
$$

Remark

- This die is much easier to predict.
- It produces much less information — less than a coin toss!
- Requires less data for storage etc.

Definition

- Function $f\colon \mathbb{R} \to \mathbb{R}$ such that

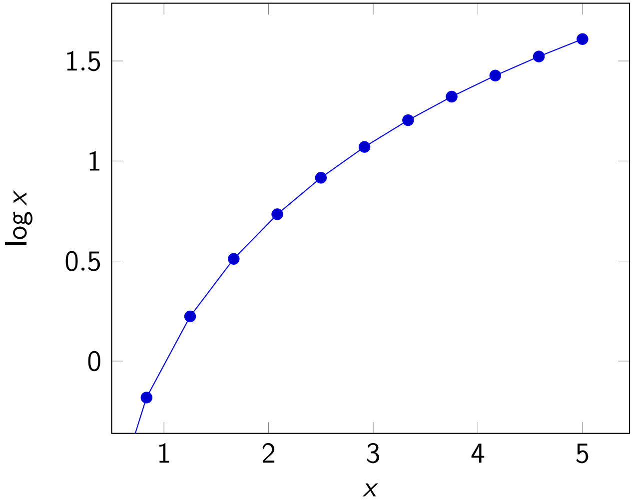
$$tf(x) + (1-t)f(y) \leq f(tx + (1-t)y),$$

- Then f is *concave*.
- With strict inequality for $x \neq y$ we say that f is *strictly concave*.

Example

$\log\colon \mathbb{R} \to \mathbb{R}$ is strictly concave.

Properties for Shannon entropy



Theorem (Jensen's inequality)

- *Strictly concave function* $f : \mathbb{R} \to \mathbb{R}$.
- *Real numbers* $a_1, a_2, \ldots, a_n > 0$ *such that* $\sum_{i=1}^{n} a_i = 1$.
- *Then we have*

$$\sum_{i=1}^{n} a_i f(x_i) \leq f\left(\sum_{i=1}^{n} a_i x_i\right).$$

- *We have equality iff* $x_1 = x_2 = \cdots = x_n$.

Theorem

- *Stochastic variable* $\mathbf{X}$ *with probability distribution*

$$p_1, p_2, \ldots, p_n, \text{ where } p_i > 0 \text{ for } 1 \leq i \leq n.$$

- *Then* $H(\mathbf{X}) \leq \log n$.
- *Equality iff* $p_1 = p_2 = \cdots = p_n = 1/n$.

Proof.

The theorem follows directly from Jensen's inequality:

$$H(\mathbf{X}) = -\sum_{i=1}^{n} p_i \log p_i = \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$$
$$\leq \log \sum_{i=1}^{n} p_i \frac{1}{p_i} = \log n.$$

With equality iff $p_1 = p_2 = \cdots = p_n$. Q.E.D.

Corollary

$H(\mathbf{X}) = 0$ iff $\Pr(\mathbf{X} = x) = 1$ for some $x \in X$ and $\Pr(\mathbf{X} = x') = 0$ for all $x \neq x' \in X$.

Proof.

- If $\Pr(\mathbf{X} = x) = 1$, then $n = 1$ and thus $H(\mathbf{X}) = \log n = 0$.
- If $H(\mathbf{X}) = 0$, then $H(\mathbf{X}) \leq \log n = 0$. Thus $n = 1$.

Q.E.D.

Lemma

- *Stochastic variables* $\mathbf{X}$ *and* $\mathbf{Y}$*.*
- *Then we have*

$$H(\mathbf{X}, \mathbf{Y}) \leq H(\mathbf{X}) + H(\mathbf{Y}).$$

- *Equality iff* $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent.*

Definition (Conditional entropy)

- Define *conditional entropy* $H(\mathbf{Y} \mid \mathbf{X})$ as

$$H(\mathbf{Y} \mid \mathbf{X}) = -\sum_{y} \sum_{x} \Pr(\mathbf{Y} = y) \Pr(\mathbf{X} = x \mid y) \log \Pr(\mathbf{X} = x \mid y).$$

Remark

This is the uncertainty in $\mathbf{Y}$ which is not revealed by $\mathbf{X}$.

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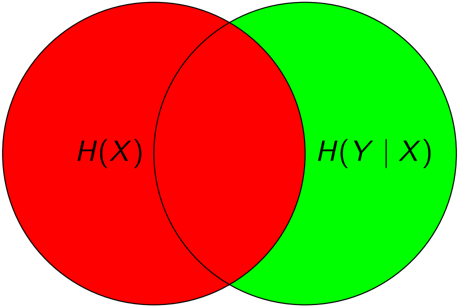
Theorem

$H(\mathbf{X}, \mathbf{Y}) = H(\mathbf{X}) + H(\mathbf{Y} \mid \mathbf{X})$



Corollary

$H(\mathbf{X} \mid \mathbf{Y}) \leq H(\mathbf{X})$.

Corollary

$H(\mathbf{X} \mid \mathbf{Y}) = H(\mathbf{X})$ *iff* $\mathbf{X}$ *and* $\mathbf{Y}$ *independent.*

Definition

- Natural language L.
- Stochastic variable $\mathbf{P}_L^n$ of strings of length n.
- Entropy of L defined as

$$H_L = \lim_{n \to \infty} \frac{H(\mathbf{P}_L^n)}{n}.$$

- Redundancy in L is

$$R_L = 1 - \frac{H_L}{\log |P_L|}.$$

Remark

Meaning we have H_L bits per character in L.

Example ([Sha48])

- Entropy of 1–1.5 bits per character in English.
- Redundancy of approximately $1 - \frac{1.25}{\log 26} = 0.61$.

Example ([Sha48])

Two-dimensional cross-word puzzles requires redundancy of approximately 0.5.

Example

- Redundancy of ‘SMS languages’ is lower than for ‘non-SMS languages’.
- Compare ‘också’ and ‘oxå’.

Remark

- Lower redundancy is more space-efficient.
- Incurs more errors.

Definition

- Set U of possible outcomes.
- Probability of outcome $u \in U$ denoted p_u.
- We learn that some *unknown* outcome is in $A \subset U$.
- Then the *information gain* $G(A \mid U)$ is defined as

$$G(A \mid U) = \log \frac{1}{\Pr(A)} = -\log \Pr(A),$$

where $\Pr(A) = \sum_{i \in A} p_i$.

Example (Roll of dice again)

- Someone rolls and we should guess the result, $\frac{1}{6}$ chance.
- We learn that it was an even number, we gain

$$-\log\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right) = -\log\frac{3}{6} = \log\frac{6}{3} = \log 2 = 1.$$

- The remaining uncertainty is 1.58 bit.

Remark

- $X' = \{⚁, ⚃, ⚅\}$
- $H(\mathbf{X}') = -\sum_{x\in X'} \Pr(\mathbf{X}' = x)\log\Pr(\mathbf{X}' = x)$
- I.e. $-3 \times \frac{1}{3}\log\frac{1}{3} \approx 1.58$.

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Example (Dice yet again)

- We learn the die show less than five, i.e. not ⚄ or ⚅.
- This yields

$$-\log\left(4 \times \frac{1}{6}\right) = \log \frac{6}{4} \approx 0.58$$

Idea [Kom+11]

- Look at different aspects of passwords individually, then summarize.
- Can use $H(x_1, x_2, \ldots, x_n) \leq H(x_1) + H(x_2) + \cdots + H(x_n)$.
- This allows us to reason about bounds.

Example

- We can look at properties such as:
  - length,
  - number of and placement of character classes,
  - the actual characters,
  - ...

Remark

- These are *not independent*.
- The sum will be an *upper bound*.

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Remark

- With an upper bound we know it's not possible to do better.
- With an average we know how well most users will do.
- With a lower bound we have a guarantee — not possible!

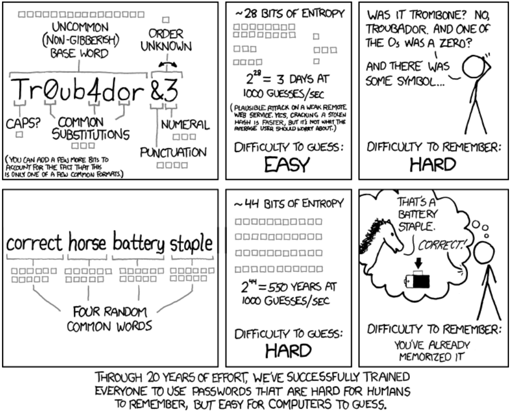

Figure: xkcd:s serie om lösenordsstyrka. Bild: xkcd [xkc].

Example (Standard password)

- We have
  - 26 alphabetic characters,
  - 10 numbers,
  - 10 special characters (approximately).
- This yields $\log(2 \times 26 + 10 + 10) = \log 72 \approx 6\,\text{bit}$ per password character.
- A 10-character *uniformly randomly* generated password contains 60 bit.

Remark

What happens when we require two upper and two lower-case characters, two numbers must be included?

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Example (Four-word passphrase)

- We have 125 000 words in the standard Swedish dictionary.
- This yields $\log 125\,000 \approx 17$ bit per word.
- A four-word *uniformly randomly* generated passphrase contains 68 bit.

Example (Random sentence)

- We estimated the entropy per character in a language.
- It was approximately 1.25 bit for English.
- A 20-character *uniformly randomly* generated sentence would yield 25 bit.

Remark

- All these require uniform randomness.
- Humans are bad at remembering random things.
- Thus they will choose non-randomly.
- The entropy will thus be (possibly much) lower.

Example ('Linguistic properties of multi-word passwords' [BS12])

- Investigates how linguistics affect the choice of multi-word passphrases.
- Users don't choose them randomly, prefer adapted to natural language.
- 'correct horse battery staple' is preferred to 'horse correct battery staple' since the first is more grammatically correct.

Example (*Human Selection of Mnemonic Phrase-based Passwords* [KRC06])

- Studied how users creates easy-to-remember passwords.
- Also investigated the strength of phrase-based passwords.
- E.g. Google's example 'To be or not to be, that is the question'[1] which results in '2bon2btitq'.
- This particular password has apparently been used by many . . .

[1] URL: `http://www.lightbluetouchpaper.org/2011/11/08/want-to-create-a-really-strong-password-dont-ask-google/`.

Remark

- There is a PhD thesis on the topic of guessing passwords: Joseph Bonneau. *Guessing human-chosen secrets*. Tech. rep. UCAM-CL-TR-819. University of Cambridge, Computer Laboratory, May 2012. URL: http://www.cl.cam.ac.uk/techreports/UCAM-CL-TR-819.pdf.
- There is even a conference dedicated to passwords: PasswordsCon.

Example

Do we get more information from zodiac signs or birthdays?

$$-\sum_{\text{stjärntecken}} \frac{1}{12} \log \frac{1}{12} = \log 12 \approx 3.58$$

$$< -\sum_{\text{dagar på året}} \frac{1}{365} \log \frac{1}{365} = \log 365 \approx 8.51.$$

Exercise

How much information do we need to uniquely identify an individual?

Example

- Sometime during 2011 there were $n = 6\,973\,738\,433^2$ people on earth.
- To give everyone a unique identifier we need $\log n = 32.7 \approx 33$ bits of information.

[2] According to the World Bank.

Identifying information in browsers

- Electronic Frontier Foundation (EFF) studied [Eck10] how much information a web-browser shares.
- You can try your browser in `http://panopticlick.eff.org/`.

Example (My browser)

- My Firefox-browser with all addons gave 21.45 bits of entropy.
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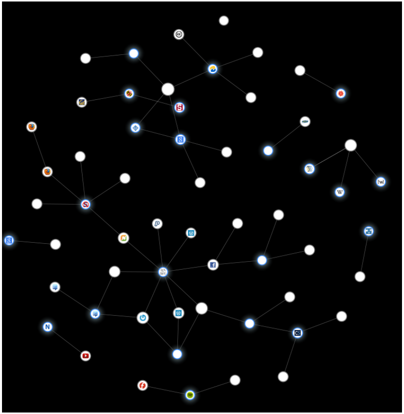

Figure: Screenshot from Collusion (now Lightbeam) for Firefox. Map over all pages that track me using this information.

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