

A High-Level Overview of Cryptography

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1 Introduction

- History
- Kerckhoff's Principle

2 Symmetric Cryptography

- Ciphers
- Security
- Hash Functions
- Message-authentication codes

3 Public-Key Cryptography

- Key-Exchange Schemes
- Encryption and Decryption
- Digital Signatures
- Homomorphic Properties

4 More Counter-Intuitive Things

- (Secure) Multi-Party Computation
- Zero-Knowledge Proofs of Knowledge



- The word has its origin in greek¹:
 - κρυπτός (*kryptos*) meaning hidden².
 - γράφος (*graphos*) meaning writing³.
- The area has been around for ages.
- We should not confuse it with *steganography*.
- Steganography concerns hiding a message's *existence*.
- Cryptography concerns hiding a message's *contents*.

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- Then it was an art, now it's a science.
- People used 'clever' constructions.
- These were thought to be secure: 'How can anyone figure this out?'
- Well, it turns out that there are always a lot of people with a lot of time and motivation . . .

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A quote⁴

[A cryptosystem] should not require secrecy, and it should not be a problem if it falls into the enemy hands;

Kerckhoff's Principle

- No security-by-obscurity
- The key should be the only secret

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Note

- This doesn't mean we must tell the adversary what we're using.
- But we shouldn't lose any security if we do.

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Idea

- Alice and Bob share a (small) common secret.
- Alice takes a message, combines it with the secret, sends it to Bob.
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Block-cipher encryption

Input A fixed-sized *key* k , a fixed-sized block of *plaintext* p .

Output A fixed-sized block of *ciphertext* c .

Notation $\text{Enc}_k(p) = c$

Block-cipher decryption

Input A fixed-sized *key* k , a fixed-sized block of *ciphertext* c .

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Definition (Crypto system⁵)

- A *crypto system* is a tuple $(\mathcal{M}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where:
 - \mathcal{M} is a finite set of *plaintexts* or messages,
 - \mathcal{C} is a finite set of *ciphertexts*,
 - \mathcal{K} is the *keyspace*, a finite set of keys.
 - \mathcal{E} and \mathcal{D} are the sets of encryption and decryption rules, respectively.
- For every $k \in \mathcal{K}$ there is a $\text{Enc}_k \in \mathcal{E}$ and $\text{Dec}_k \in \mathcal{D}$ such that
 - $\text{Enc}_k: \mathcal{M} \rightarrow \mathcal{C}$ and $\text{Dec}_k: \mathcal{C} \rightarrow \mathcal{M}$ are functions and
 - $\text{Dec}_k(\text{Enc}_k(m)) = m$ for all plaintexts $m \in \mathcal{M}$.

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Definition (Shift Cipher)

- Let $\mathcal{M} = \mathcal{C} = \mathcal{K} = \mathbb{Z}_{29}$
- For each $k \in \mathcal{K}$ we define

$$\text{Enc}_k(m) = (m + k) \bmod 29, m \in \mathcal{M}, \text{ och}$$

$$\text{Dec}_k(c) = (c - k) \bmod 29, c \in \mathcal{C}.$$

Example

- $\text{Enc}_3(7) = 7 + 3 \bmod 29 = 10$ h → J
- $\text{Enc}_3(4) = 4 + 3 \bmod 29 = 7$ e → G
- $\text{Enc}_3(9) = 9 + 3 \bmod 29 = 12$ j → L

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Note

- The shift cipher is a classical cipher — also known as the Caesar Cipher.
- It's easily broken *by hand*!
- It's used here for illustrative purposes.

Exercise

- What do we have to do to set this up between two parties, say Alice and Bob?
- What problems do we have to solve?

Definition (Perfect secrecy⁶)

- Cryptosystem $(\mathcal{M}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$.
- Stochastic variables M, C .
- *Perfect secrecy* if and only if

$$\Pr(M = m \mid C = c) = \Pr(M = m)$$

for all $m \in \mathcal{M}$ and $c \in \mathcal{C}$.

Note

Equivalent to $H(M \mid C) = H(M)$, i.e. ciphertext does not reveal anything about plaintext.

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Theorem (Shannon's theorem)

- Assume cryptosystem $(\mathcal{M}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$ such that $|\mathcal{K}| = |\mathcal{C}| = |\mathcal{M}|$.
- This provides perfect secrecy if and only if
 - 1 every key $k \in \mathcal{K}$ is used with equal probability $1/|\mathcal{K}|$,
 - 2 for every plaintext $m \in \mathcal{M}$ and $c \in \mathcal{C}$ there is a unique key such that $\text{Enc}_k(m) = c$.

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Example (One-time Pad)

- Let n be a positive integer.
- Let $\mathcal{M} = \mathcal{C} = \mathcal{K} = (\mathbb{Z}_2)^n$.
- For each key $k = (k_1, \dots, k_n) \in \mathcal{K}$, plaintexts $m = (m_1, \dots, m_n) \in \mathcal{M}$ and ciphertexts $c = (c_1, \dots, c_n) \in \mathcal{C}$ we define

$$\text{Enc}_k(m) = (m_1 + k_1, \dots, m_n + k_n)$$

- We also define $\text{Dec} = \text{Enc}$.
- $k \in \mathcal{K}$ must be chosen uniformly randomly for each encryption.

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Definition (Pseudo-random permutation, PRP⁷)

- Let $F: \{0, 1\}^s \times \{0, 1\}^n \rightarrow \{0, 1\}^n$.
- F is a PRP if
 - 1 for any $k \in \{0, 1\}^s$, F is a bijection;
 - 2 for any $k \in \{0, 1\}^s$, we can 'efficiently' evaluate $F_k(x)$;
 - 3 for all 'efficient' distinguishers D ,

$$|\Pr[D^{F_k}(1^n) = 1] - \Pr[D^{f_n}(1^n) = 1]| < \epsilon(s)$$

if we choose $k \in \{0, 1\}^s$ and the random permutation f_n uniformly at random.

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Idea

- We want a function which we can efficiently compute.
- However, it shouldn't be possible to find its inverse.

Example

Easy $f(x) = y$

Hard $f^{-1}(y) = x$



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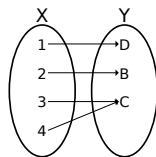
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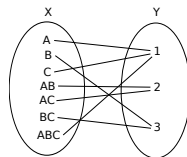
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Hash Functions



(a)
 $h: X \rightarrow Y$



(b) $h': X \rightarrow Y$

Figure: Two non-injective, surjective functions h and h' .

Exercise

Could either of these two functions be one-way functions?

Definition (One-way function⁸)

- Let $h: \{0, 1\}^* \rightarrow \{0, 1\}^*$.
- h is *one-way* if
 - 1 there exists an efficient algorithm A such that $A(x) = h(x)$;
 - 2 for every efficient algorithm A' , every positive polynomial $p(\cdot)$ and all sufficiently large n 's

$$\Pr[A'(h(x), 1^n) \in h^{-1}(h(x))] < \frac{1}{p(n)}$$

⁸Oded Goldreich. *Foundations of cryptography, Vol. 1: Basic tools*.

Cambridge: Cambridge Univ. Press, 2001. ISBN: 0-521-79172-3.

Example (Implementations you might've heard of)

- MD5
- SHA1
- SHA256 (SHA-2)
- SHA-3

Example (Applications)

- Verifying file content integrity
- Digital signatures
- Protect passwords

Note

- One-wayness returns as a useful property in many situations.
- Encryption also has the one-wayness property:
 - Easy** Given k, m , compute $c \leftarrow \text{Enc}_k(m)$.
 - Hard** Given c , compute either of k, m .
- However, encryption is bijective, hash functions are generally not.

Example

- Let $\text{Enc}_k(\cdot) = \text{Dec}_k(\cdot) = \cdot \oplus k \pmod 2$.
- Alice and Bob share k .
- Alice sends $\text{Enc}_k(m) = c$ to Bob.
- Eve intercepts c , she cannot get to m .
- Eve computes $c' = c \oplus m_E$ and passes c' to Bob.
- Bob computes

$$\text{Dec}_k(c') = \text{Dec}_k(c \oplus m_E) = m \oplus k \oplus m_E \oplus k = m \oplus m_E.$$

Exercise

How can we solve this? Bob needs to know that Eve modified the message!

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Idea: MACs

- Alice and Bob need something that Eve doesn't know how to modify.
- If that something is tied to the message, then a modified message would be detectable.

Exercise

Any ideas on how we can construct such a thing?

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Example

- Let h be a one-way function.
- If we use $h(c) = t$, then Eve can also compute the hash function: $h(c') = t'$.
- A secret hash function would violate Kerckhoff's principle, so that's not an option.
- If we instead use the message, rather than the ciphertext.
- Then $h(m) = t$ and
 - $\text{Dec}_k(c') = m' = m \oplus m_E, h(m') \neq t.$
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- Eve cannot compute the hash function, she doesn't have $m!$
 - Bob: But neither do I!

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- A secret hash function would violate Kerckhoff's principle, so that's not an option.
- If we instead use the message, rather than the ciphertext.
- Then $h(m) = t$ and
 - $\text{Dec}_k(c') = m' = m \oplus m_E, h(m') \neq t.$
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Solution

- *Let s be a secret shared between Alice and Bob.*
- *$h(c \parallel s) = t$, Eve doesn't know s .*
- *Bob can immediately check $h(c' \parallel s) \neq t$.*

Note

- It requires even a bit more than this!
- But the idea is correct.

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Solution (Hash-based message-authentication code, HMAC⁹)

- *Let h be a one-way function.*
- *Let c be the ciphertext, s our MA secret.*
- *Then tag $t = \text{HMAC}_s(c)$, where*

$$\text{HMAC}_s(c) = h[(s \oplus p_o) \parallel h[(s \oplus p_i) \parallel c]],$$

and p_i, p_o are inner and outer pads, respectively.

Note

This is proven secure in [9]!

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Idea

- It's difficult to have to exchange keys in advance.
- What if we could securely exchange keys at a distance?
- If we could do it just before we use them?

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- What if we could securely exchange keys at a distance?
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Solution (Requirements)

- *We need a problem that is easy for Alice and Bob.*
- *It should be hard for Eve.*



Definition (Discrete Logarithm Problem, DLP)

- Let \mathbb{Z}_p^* be the multiplicative group of residues modulo $p \in \mathbb{N}$, where p is a prime.

Given $g, g^x \in \mathbb{Z}_p^*$

Find x .

- I.e. compute $\log_{g \in \mathbb{Z}_p^*}(g^x)$.



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Given $g, g^x, g^y \in \mathbb{Z}_p^*$

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Given $g, g^x, g^y, g^z \in \mathbb{Z}_p^*$

Decide $z \stackrel{?}{=} xy$

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Exercise

- Diffie and Hellman¹¹ used DHP to create a key-exchange protocol.
- Take some time to figure out how we can use these problems to achieve what we want.

Reminder

- Alice and Bob want to exchange a secret key.
- Then they can use the key to encrypt their communications.

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- Let $g \in \mathbb{Z}_p^*$ (publicly known, e.g. RFC, standard dots).
- Alice generates random $0 < x < |\mathbb{Z}_p^*|$.
- She send g^x to Bob.
- Bob generates random $0 < y < |\mathbb{Z}_p^*|$.
- He sends g^y to Alice.
- Alice has x and g, g^y .
- Bob has g, g^x and y .
- They both compute $g^{xy} = (g^y)^x = (g^x)^y$.
- Eve has g, g^x, g^y .
- By DHP she cannot compute g^{xy} .

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Idea

- Fine, we can use g^{xy} as a key in a cipher.
 - $\text{Enc}_{g^{xy}}(m)$, where Enc is a symmetric cipher.
- But shouldn't we be able to include a message directly?



Definition (ElGamal Encryption Scheme¹²)

Set-up:

- Let $g \in \mathbb{Z}_p^*$, randomly choose $0 < x < |\mathbb{Z}_p^*|$.
- Alice publishes \mathbb{Z}_p^*, g, g^x to everyone.

Encryption:

- Bob chooses random $0 < y < |\mathbb{Z}_p^*|$ and computes g^y .
- Bob's message $m \in \mathbb{Z}_p^*$.
- He sends $(g^y, m(g^x)^y)$ to Alice.

Decryption:

- Alice computes $(g^y)^x$ and $m(g^x)^y((g^y)^x)^{-1} = m$.

¹²Taher ElGamal. 'A Public Key Cryptosystem and a Signature Scheme Based on Discrete Logarithms'. In: *Advances in Cryptology: Proceedings of CRYPTO 84*. Ed. by George Robert Blakley and David Chaum. Berlin, 1985. 385-405.

Idea

- Sure, if Bob sends a message to Alice, he's sure she's the only one who can decrypt it.
- Can't we turn this around?
 - Can't Alice use the same system to ensure Bob knows the message came from Alice?

Exercise

- Look at the ElGamal encryption scheme for a bit.
- Try to find a way to 'run it backwards'.



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Definition (ElGamal Signature Scheme¹³)

Set-up:

- Let $g \in \mathbb{Z}_p^*$ and h be a one-way function.
- Alice publishes \mathbb{Z}_p^* , g , g^x to everyone.

Signing $m \in \mathbb{Z}_p^*$:

- Alice chooses random $0 < y < |\mathbb{Z}_p^*|$ and computes $r = g^y \in \mathbb{Z}_p^*$.
- She computes $s = (h(m) - xr)y^{-1} \pmod{|\mathbb{Z}_p^*|}$.
- She sends (r, s) to Bob.

Verification:

- Bob checks if $g^{h(m)} \stackrel{?}{=}_{\mathbb{Z}_p^*} (g^x)^r r^s =_{\mathbb{Z}_p^*} (g^x)^r (g^y)^s$

$$(g^x)^r (g^y)^s = (g^x)^r (g^y)^{(h(m) - xr)y^{-1}} = (g^x)^r (g^y)^{h(m) - xr} = (g^x)^r (g^y)^{h(m)} (g^y)^{-xr} = (g^x)^r (g^y)^{h(m)} (g^x)^{-r} = (g^y)^{h(m)}$$



Note

- It works without the hash.
- But then we can multiply two messages and still get a valid signature.

Definition (Homomorphism)

A *homomorphism* is a map (function) that preserves structure between two algebraic structures.

Example

- Let $G_1 = (\mathbb{R}, \cdot)$ and $G_2 = (\mathbb{R}, +)$ be groups.
- $g_1, g'_1 \in G_1$ and $g_2, g'_2 \in G_2$.
- Consider $\log: G_1 \rightarrow G_2$.
- $\log(g_1 \cdot g'_1) = g_2 + g'_2$.

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Exercise

The encryption (decryption) function of the ElGamal cryptosystem is a homomorphism, what structure does it preserve?

Example (ElGamal's homomorphism)

- Messages m, m' , ciphertexts $(g^y, m \cdot g^{xy}), (g^{y'}, m' \cdot g^{xy'})$.
- Remember: private key x , hence the same.
- Create ciphertext

$$\begin{aligned} (g^y g^{y'}, m \cdot g^{xy} \cdot m' \cdot g^{xy'}) &= (g^{y+y'}, m \cdot m' \cdot g^{xy+xy'}) \\ &= (g^{y+y'}, m \cdot m' \cdot g^{x(y+y')}). \end{aligned}$$

- Decryption: take $g^{y+y'}$, compute $(g^{y+y'})^x = g^{x(y+y')}$.
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Note

- We use a hash function in the signature scheme to counter the homomorphic property.
- $h(m) \cdot h(m') \neq h(m \cdot m')$.
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Note

- There are many schemes with different homomorphic properties.
- There is even *fully homomorphic encryption* [12].

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Example (Yao's millionaires' problem)

- Two millionaires meet in the street.
- They want to find out who is the richer.
- However, they don't want to reveal how many millions they each have.

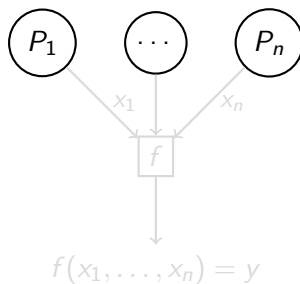
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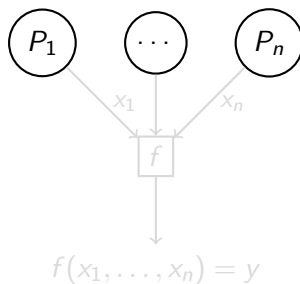
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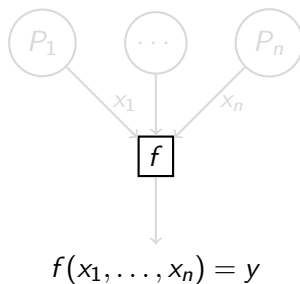
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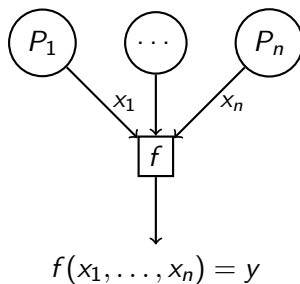




(Secure) Multi-Party Computation

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Example (Trivial solution)

- The n participants P_1, \dots, P_n agree on a trusted third-party (a TTP).
- Each participant give their secret to the TTP.
- The TTP trusted third-party performs the computation.
- Every participant receives the result from the TTP.

Definition (Secure multiparty computation, MPC)

- n participants P_1, \dots, P_n .
- n secret inputs x_1, \dots, x_n .
- A protocol π is executed by the participants.
- At the end of the protocol each participant learns $y = f(x_1, \dots, x_n)$.
- The participants executing π should be *equivalent* to giving x_1, \dots, x_n to a TTP T who computes $f(x_1, \dots, x_n) = y$ and returns y to each participant.

Note

Each participant P_i learns no more about x_j ($i \neq j$) than what is revealed by y .

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(Secure) Multi-Party Computation

- In general this problem is solved.
- We can construct protocols for arbitrary functions f .
- Efficiency varies though.
- However, there are practically feasible protocols.
- Sometimes we can use homomorphisms.
- But we can construct rather complex functions too.

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(Secure) Multi-Party Computation

Example (Sugar beet auctions¹⁴)

- Several thousand farmers produce sugar beets.
- These are sold to the monopoly Danisco, the sugar producer.
- Contracts are allocated via a nation-wide exchange, a double auction.
- A double auction contains multiple sellers and multiple buyers.
- The purpose is to find the *market clearing price*.

¹⁴Peter Bogetoft et al. 'Secure Multiparty Computation Goes Live'. In: *Financial Cryptography and Data Security: FC 2009*. Ed. by Roger Dingledine and Philippe Golle. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 325–343. ISBN: 978-3-642-03549-4. DOI: 10.1007/978-3-642-03549-4_20. URL: http://dx.doi.org/10.1007/978-3-642-03549-4_20.



Example (Sugar beet auctions¹⁴)

- Several thousand farmers produce sugar beets.
- These are sold to the monopoly Danisco, the sugar producer.
- Contracts are allocated via a nation-wide exchange, a double auction.
- A double auction contains multiple sellers and multiple buyers.
- The purpose is to find the *market clearing price*.

¹⁴Peter Bogetoft et al. 'Secure Multiparty Computation Goes Live'. In: *Financial Cryptography and Data Security: FC 2009*. Ed. by Roger Dingledine and Philippe Golle. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 325–343. ISBN: 978-3-642-03549-4. DOI: [10.1007/978-3-642-03549-4_20](https://doi.org/10.1007/978-3-642-03549-4_20). URL: http://dx.doi.org/10.1007/978-3-642-03549-4_20.

Example (Sugar beet auctions, continued)

- Each buyer places a bid specifying how much he is willing to buy *at each potential price*.
- Each seller says how much they are willing to sell at each given price.
- The auctioneer computes the total supply and demand for each price.
- We want to find where supply equals demand.
- When done, anyone who specified non-zero for this price may trade at this price.

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Example

- Alice must prove her identity to Eve.
- Eve has Alice's public key, and knows it belongs to Alice.
- Alice wants to prove she is the owner of the private key belonging to the public key that Eve has.
- Eve asks Alice to sign the message m , if the signature verifies under the public key Eve believes Alice.

Gaaahh!

- Now Eve can show this message (chosen by Eve) with Alice's signature on it!
- What if Eve's chosen message was 'I give all my money to Eve'?

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Idea

- Alice wants to prove that she knows the discrete logarithm x of a value g^x .
- She will do this without revealing x to Eve.



Definition (Schnorr's protocol¹⁵)

- Prover wants to prove knowledge of x for $g^x = y$.
- Prover commits to randomness r , by sending $t = g^r$.
- Verifier replies with randomly chosen challenge c .
- After receiving c , prover replies with $s = r + cx$.
- Verifier accepts if $g^s = g^{r+cx} = g^r(g^x)^c = ty^c$.

¹⁵C. P. Schnorr. 'Efficient signature generation by smart cards'. In: *Journal of Cryptology* 4.3 (1991), pp. 161–174. ISSN: 1432-1378. DOI: 10.1007/BF00196725. URL: <http://dx.doi.org/10.1007/BF00196725>.



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Proof outline.

- We need to prove *completeness*: for all (most) statements the verifier will accept.
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Zero-knowledge

- Transcript for protocol: (t, c, s) .
- Probability for transcript occurring: $\frac{1}{|R|} \cdot \frac{1}{\deg g}$.
- Simulate protocol: randomly choose c , randomly choose s , compute t by $g^s y^c$.
- We see that we get the same probability distribution.
- Thus the simulated transcripts are indistinguishable from the real ones.

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Zero-Knowledge Proofs of Knowledge

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